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LEVEL II

SOME GENERALIZATIONS OF KANTOROVICH INEQUALITY*

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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SOME GENERALIZATIONS OF KANTOROVICH INEQUALITY

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Summary: Kantorovich gave an upper bound to $(x'Vx)(x'V^{-1}x)$ where x is an n-vector of unit length and V is an $n\times n$ positive definite matrix. Bloomfield, Watson and Knott found the bound to $|X'VXX'V^{-1}X|$, and Khatri and Rao to the trace and determinant of $X'VYY'V^{-1}X$ where X and Y are $n\times k$ matrices such that $X'X = Y'Y = I$. In the present paper we establish bounds for traces and determinants of $X'VYY'V^{-1}X$ and $X'BYY'CX$ when X and Y are matrices of different orders. A review of previous results on generalizations of Kantorovich inequality and a number of new results of independent interest are also given.

Key Words: Kantorovich inequality, Least squares estimation.

1. INTRODUCTION

Let V be an $n \times n$ positive definite matrix with eigen values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$, and define $\omega_i = (\lambda_i + \lambda_{n+i-1})^2/4\lambda_i\lambda_{n+i-1}$, $i = 1,\ldots,\alpha(\leq n/2)$. Kantorovich (1948) established the inequality

$$1 \leq (x'Vx)(x'V^{-1}x)/(x'x)^2 \leq \omega_1 \tag{1.1}$$

for all non-null vectors x. A natural generalization of (1.1) is to compare the matrices $A = (X'X)^{-1}X'VX(X'X)^{-1}$ and $B = (X'V^{-1}X)^{-1}$ when X varies over $n \times k$ matrices of rank k. Let $\theta_1,\ldots,\theta_k$ be the roots of $|A - \theta B| = 0$, i.e., the eigen values of A with respect to B. It is easy to establish that $\theta_i \geq 1$ for all i. Bloomfield and Watson (1975) and Knott (1975) showed that

$$|AB^{-1}| = \theta_1,\ldots,\theta_k \leq \prod_{i=1}^{\min(k,n-k)} \omega_i \tag{1.2}$$

while Khatri and Rao (1980) established that

$$\begin{aligned} \operatorname{tr}(AB^{-1}) = \theta_1+\ldots+\theta_k &\leq \omega_1+\ldots+\omega_k \quad \text{if } n \geq 2k \\ &\leq \omega_1+\ldots+\omega_{n-k} + (2k-n) \quad \text{if } n < 2k \end{aligned} \tag{1.3}$$

where $\operatorname{tr} C$ represents the trace of matrix C. It may be noted that $\Sigma\,\theta_i = \operatorname{tr} P_X V P_X V^{-1}$ where P_X is the projection operator on the column space of X. Bloomfield and Watson (1975) gave another inequality

$$\operatorname{tr} P_X V (I - P_X) V \leq \frac{1}{4} \sum_{1}^{k} (\lambda_i - \lambda_{n-i+1})^2 \tag{1.4}$$

when $n \geq 2k$. The inequalities (1.2) - (1.4) are useful in comparing the efficiencies of simple least squares estimators with the minimum variance unbiased estimators of parameters in the Gauss-Markoff model (see Khatri and Rao, 1980).

Khatri (1978, 1980) considered the matrix $(I - B A^{-1})$ which arises in a different context and proved the following results. Let $\underset{\sim}{i} = (i_1, i_2, \ldots, i_n)$ be a permutation of $(1,2,\ldots,n)$ and P denote the class of all permutations of $(1,2,\ldots,n)$. Further, let

$$\xi_{\underset{\sim}{i}(\alpha)} = (\lambda_{i_\alpha} - \lambda_{i_{n-\alpha+1}})^2 / (\lambda_{i_\alpha} + \lambda_{i_{n-\alpha+1}})^2 \tag{1.5}$$

for $\alpha = 1,2,\ldots,k$ with $n \geq 2k$. Then

$$|I - B A^{-1}| \leq \sup_{\underset{\sim}{i} \in P} \prod_{\alpha=1}^{\min(k,n-k)} \xi_{i(\alpha)} = \prod_{j=1}^{\min(k,n-k)} \frac{(\lambda_j - \lambda_{n-j+1})^2}{(\lambda_j + \lambda_{n-j+1})^2}, \tag{1.6}$$

$$\operatorname{tr} (I - B A^{-1}) \leq \sup_{\underset{\sim}{i} \in P} \sum_{\alpha=1}^{k} \xi_{i(\alpha)} \quad \text{if} \quad n \geq 2k, \tag{1.7}$$

$$\operatorname{tr} (I - B A^{-1})^{-1} \geq \inf_{i \in P} \sum_{\alpha=1}^{k} \xi^{-1}_{\underset{\sim}{i}(\alpha)} \quad \text{if} \quad n \geq 2k. \tag{1.8}$$

Khatri (1978) further showed that if

$$C = X' V Y (Y' V Y)^{-1} Y' V X (X' V X)^{-1} \tag{1.9}$$

where Y is an $n \times s$, $(s \leq n-k)$, matrix such that $X'Y = 0$, then

$$|C| \leq |Q|, \ \mathrm{tr}\ C \leq \mathrm{tr}\ Q, \text{ and } \mathrm{tr}\ C^{-1} \geq \mathrm{tr}\ Q^{-1} \qquad (1.10)$$

where $Q = I - BA^{-1}$. Eaton (1976) established the results (1.6) and (1.10) when $k=s=1$.

Strang (1960) generalized the Kantorovich inequality (1.1) in the form

$$[(x'Ay)(y'A^{-1}x)/(x'x)(y'y)] \leq \omega_1 \qquad (1.11)$$

for all non-null vectors x and y where A is an $n \times n$ nonsingular matrix with singular values $\delta \geq \delta_2 \geq \ldots \geq \delta_n > 0$ and

$$\omega_i = (\delta_i + \delta_{n-i+1})^2/4\ \delta_i\ \delta_{n-i+1}. \qquad (1.12)$$

Greub and Rheinboldt (1959) proved that

$$\frac{(x'G^2x)(x'H^2x)}{(x'GHx)^2} \leq \frac{(\lambda_1\mu_1+\lambda_n\mu_n)^2}{4\ \lambda_1\lambda_n\mu_1\mu_n} \qquad (1.13)$$

for all non-null vectors x, when G and H are positive definite commuting matrices with eigen values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > C$ and $\mu_1 \geq \mu_2 \geq \ldots \geq_n > 0$ respectively. The result (1.13) can be proved using (1.11).

A natural generalization of the expression in (1.11) is of the form

$$g(X,Y) = |X' A P_Y A^{-1} X| / |X' X| \tag{1.14}$$

or

$$f(X,Y) = \operatorname{tr}(P_X A P_Y A^{-1}) \tag{1.15}$$

where X and Y are $(n \times k)$ and $(n \times s)$ matrices of ranks k and s respectively with $s \geq k$, and A is an $n \times n$ non-singular matrix with singular values $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$ 0. When $s = k$, Khatri and Rao (1980) established the inequalities

$$g(X,Y) \leq \prod_{i=1}^{\min(k,n-k)} \omega_i \tag{1.16}$$

$$\begin{aligned} f(X,Y) &\leq \sum_{i=1}^{k} \omega_i \quad \text{if} \quad n \geq 2k \\ &\leq \sum_{i=1}^{n-k} \omega_i + 2k - n \quad \text{if} \quad n < 2k \end{aligned} \tag{1.17}$$

where ω_i are as defined in (1.12). In the next section we establish the bounds for (1.14) and (1.15) when s is not necessarily equal to k.

We also consider determinants and traces of matrices of the type

$$(X' G^2 Y)(Y' G H Y)^{-1}(Y' H^2 X)(X' G H X)^{-1} \tag{1.18}$$

which are natural extensions of the expression in (1.13) and establish bounds under very general conditions on X, Y, G and H.

2. THE MAIN THEOREMS

In all the theorems stated in this section, X and Y stand for $n \times k$ and $n \times s$ matrices of ranks k and s respectively with $s \geq k$ and P_Z stands for the projection operator on the column space of matrix Z.

Theorem 1. Let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n > 0$ be the singular values of an $n \times n$ nonsingular matrix A, and

$$\omega_i = \frac{(\delta_i + \delta_{n-i+1})^2}{4\delta_i \ \delta_{n-i+1}}, \ i = 1, \ldots, m(\leq n/2). \qquad (2.1)$$

Then

$$\frac{|X' A P_Y A^{-1} X|}{|X'X|} \leq \prod_{i=1}^{\min(k,n-s)} \omega_i \qquad (2.2)$$

and

$$\operatorname{tr}(P_X A P_Y A^{-1}) \leq \sum_{i=1}^{k} \omega_i \quad \text{if} \quad n \geq s+k$$

$$\leq \sum_{i=1}^{n-s} \omega_i + (s+k-n) \quad \text{if} \ n < s+k \qquad (2.3)$$

Note 1. If X_1 and Y_1 are $n \times (n-k)$ and $n \times (n-s)$ matrices which are orthogonal complements of X and Y respectively, then $(n-k) \geq (n-s)$ and

$$\frac{|X' A P_Y A^{-1} X|}{|X'X|} = \frac{|Y_1' A^{-1} P_{X_1} A \ Y_1|}{|Y_1' \ Y_1|} \qquad (2.4)$$

and

$$\mathrm{tr}\,(P_X A P_Y A^{-1}) = (s+k-n) + \mathrm{tr}\,(P_{Y_1} A^{-1} P_{X_1} A). \qquad (2.5)$$

The results (2.4) and (2.5) show that we need only consider the case $s \geq k$ and $n \geq s+k$ in proving Theorem 1. If $n < s+k$, then $n \geq (n-s)+(n-k)$ in which case we consider the expressions on the right-hand sides of (2.4) and (2.5) and apply the same proof.

Note 2. If $A = P D_\delta Q'$ is the singular value decomposition of A, then we can write the left-hand side expressions of (2.2) and (2.3) as

$$|X_*' D_\delta Y_* Y_*' D_\delta^{-1} X_*| \text{ and } \mathrm{tr}\,(X_*' D_\delta Y Y_*' D_\delta^{-1} X_*) \qquad (2.6)$$

choosing $X_* = P'X(X'X)^{-\frac{1}{2}}$ and $Y_* = Q'Y(Y'Y)^{-\frac{1}{2}}$ so that $X_*' X_* = I_k$ and $Y_*' Y_* = I_s$, and $D_\delta = \mathrm{Diag}(\delta_1, \ldots, \delta_n)$ (i.e., a diagonal matrix with $\delta_1, \ldots, \delta_n$ as diagonal elements). In view of (2.6), we need only prove Theorem 1 with the restrictions $X'X = I_k$, $Y'Y = I_s$ and A is a diagonal matrix.

Theorem 2. Let X and Y be $n \times k$ and $m \times s$ matrices of ranks k and s respectively with $k \leq s$, and B and C be $n \times m$ and $m \times n$ matrices such that $C = B^+$ (the Moore-Penrose inverse of B, see Rao, 1973, p. 26). Further, let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_t > 0$ be the nonzero singular values of B and $t \geq s$. Then

$$\frac{|X'B P_Y C X|}{|X'X|} \leq \prod_{i=1}^{\min(k,t-s)} \omega_i \qquad (2.7)$$

and

$$\operatorname{tr}(P_X B P_Y C) \leq \Sigma\, \omega_i \quad \text{if } t \geq s+k$$

$$\leq s+k-t+\sum_{i=1}^{t-s} \omega_i \quad \text{if } t < s+k \qquad (2.8)$$

where $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4\delta_i\, \delta_{t-i+1}$.

<u>Theorem 3</u>. Let V and W be $n \times n$ and $m \times m$ non-negative definite matrices, and X and Y be $n \times k$ and $m \times s$ matrices such that $X'VX$ and $Y'VY$ are positive definite. Further, let B and C be $n \times m$ and $m \times n$ matrices such that

(a) $t = R(B) = R(C) \leq s$

(b) $\rho(W) = \rho\binom{B}{W} = \rho\,(C:V)$, $\rho(V) = \rho(B:V) = \rho\binom{C}{V}$

where $\rho(A)$ stands for the rank of matrix A.

(c) BW^+C and CV^+B are symmetric of rank t and $BW^+CV^+B = B$.

If $\delta_1^2 \geq \delta_2^2 \geq \ldots \geq \delta_t^2 > 0$ are the nonzero eigen values of $BW^+B'V^+$ and $\omega_i = (\delta_i + \delta_{t-i+1})^2/4\,\delta_i\,\delta_{t-i+1}$, then

$$\frac{|X'BY(Y'WY)^{-1}Y'CX|}{|X'VX|} \leq \prod_{i=1}^{\min(k,t-s)} \omega_i \qquad (2.9)$$

and

$$\operatorname{tr}[(X'VX)^{-1}X'BY(Y'WY)^{-1}Y'CX] \leq \sum_{i=1}^{k} \omega_i, \text{ if } t \geq s+k$$

$$\leq (s+k-t) + \sum_{i=1}^{t-s} \omega_i, \text{ if } t < s+k. \qquad (2.10)$$

Theorem 4. Let S and R be $n \times m$ and $m \times n$ matrices such that $t = \rho(S) = \rho(R) = \rho(SR)$ with $t \geq s \geq k$, and SR and RS are symmetric, nonnegative definite and idempotent matrices. Further let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_t > 0$ be the nonzero singular values of S and $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4\delta_i \delta_{t-i+1}$. Then

$$\frac{|X' S Y(Y'RSY)^{-1} Y' RX|}{|X'SRX|} \leq \prod_{i=1}^{\min(k,t-s)} \omega_i \tag{2.11}$$

and

$$\begin{aligned} \operatorname{tr}[(X'SRX)^{-1} X' SY(Y'RSY)^{-1} Y'RX] & \\ &\leq \sum_{i=1}^{k} \omega_i \quad \text{if } t \geq s+k \\ &\leq (s+k-t) + \sum_{i=1}^{t-s} \omega_i \quad \text{if } t < s+k. \end{aligned} \tag{2.12}$$

Proof. Theorem 4 follows from Theorem 3 by choosing $V = SR$, $W = RS$, $B = S$ and $C = R$.

Theorem 5. Let G and H be symmetric and commuting matrices such that GH is nonnegative definite and $\rho(G) = \rho(H) = \rho(GH) = t \geq s \geq k$. Further let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_t > 0$ be the nonzero eigen values of $H^- G$ where H^- is any g-inverse of H, and $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4\delta_i \delta_{t-i+1}$. Then

$$\frac{|X'G^2 Y(Y'GHY)^{-1} Y' H^2 X|}{|X' GHX|} \leq \prod_{i=1}^{\min(k,t-s)} \omega_i \tag{2.13}$$

and

$$\operatorname{tr}[(X'GHX)^{-1}X'G^2Y(Y'GHY)^{-1}Y'H^2X]$$

$$\leq \sum_{i=1}^{k} \omega_i \quad \text{if} \quad t \geq k+s$$

$$\leq (k+s-t) + \sum_{i=1}^{t-s} \omega_i \quad \text{if} \quad t < k+s. \tag{2.14}$$

<u>Proof</u>. Theorem 5 follows from Theorem 3 by choosing $V = W = GH$, $B = G^2$ and $C = H^2$.

<u>Theorem 6</u>. Let V and W be positive definite, and B and C be two matrices such that $\rho(B) = \rho(C) = t(\geq s)$, $BW^{-1}C$ and $CV^{-1}B$ are symmetric and $BW^{-1}CV^{-1}B = B$. Further let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_t > 0$ be the nonzero singular values of $V_1^{-1}B(W_1')^{-1}$ where $W = W_1W_1'$ and $V = V_1V_1'$. Then the inequalities (2.9) and (2.10) hold.

3. PROOFS OF MAIN THEOREMS

<u>Proof of Theorem 1</u>. The proof depends on a number of lemmas which are also of independent interest. As observed in notes 1 and 2 following the statement of Theorem 1, we can take X and Y such that $X'X = I_k$, $Y'Y = I_s$, $n \geq s+k$ and A as $D_\delta = \operatorname{Diag}(\delta_1, \ldots, \delta_n)$ with all δ_i positive.

<u>Lemma 1</u>. Let V be an n.n.d. matrix or order n with non-zero eigen values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$ and X be an $n \times k$ matrix of rank $k\,(\leq s)$. Then

$$\sup_X \frac{|X'VX|}{|X'X|} = \prod_{i=1}^{k} \lambda_i \quad \text{and} \quad \sup_X \operatorname{tr}(P_X V) = \sum_{i=1}^{k} \lambda_i \tag{3.1}$$

Proof. The result (3.1) is an immediate consequence of the Poincaré Separation Theorem (see Rao, 1979, p. 364) which states that

$$\mu_i \leq \lambda_i,\ i = 1,\ldots,k\ (\leq s) \tag{3.2}$$

where μ_i, $i = 1,2,\ldots$ are the roots of the determinantal equation

$$|X'VX - \mu X'X| = 0 \tag{3.3}$$

and the equality in (3.2) is attained for a suitably chosen X.

Lemma 2. Let X and Y be $n \times k$ and $n \times s$ matrices such that $X'X = I_k$, $Y'Y = I_s$ and $s \geq k$. Let D_δ be a positive definite diagonal matrix and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$ be the eigen values of $(Y'D_\delta Y)(Y'D_\delta^{-1} Y)$. Then

$$|X'D_\delta Y Y'D_\delta^{-1} X| \leq \left(\prod_{i=1}^{k} \lambda_i\right)^{\frac{1}{2}} |X'D_\delta X X'D_\delta^{-1} X|^{\frac{1}{2}} \tag{3.4}$$

and

$$\operatorname{tr}(P_X D_\delta P_Y D_\delta^{-1}) \leq \left(\sum_{i=1}^{k} \lambda_i\right)^{\frac{1}{2}} (\operatorname{tr} P_X D_\delta P_X D_\delta^{-1})^{\frac{1}{2}}. \tag{3.5}$$

Proof of (3.4). Consider

$$|X' D_\delta Y Y' D_\delta^{-1} X|^2 = |(D X)' B (D X)|$$

where $D^2 = D_\delta$ and

$$B = D Y Y' D_\delta^{-1} X X' D_\delta^{-1} Y Y' D$$

which is an n.n.d. matrix of rank k. If $\alpha_1, \ldots, \alpha_k$ are the nonzero eigen values of B, then

$$\left(\prod_{i=1}^{k} \alpha_i\right) = |X' D_\delta^{-1} Y Y' D_\delta^{-1} X| . \tag{3.6}$$

Hence using the result (3.1) of Lemma 1 with X as $D X$ and V as B we have

$$|X' D_\delta Y Y' D_\delta^{-1} X|^2 \leq |X' D_\delta X| \cdot | X' D_\delta^{-1} Y Y' D_\delta Y Y' D_\delta^{-1} X| . \tag{3.7}$$

Now,

$$|X' D_\delta^{-1} Y Y' D_\delta Y Y' D_\delta^{-1} X| = |(D^{-1} X)' C (D^{-1} X)| \tag{3.8}$$

where $C = D^{-1} Y Y' D_\delta Y Y' D^{-1}$ is an n.n.d. matrix of rank s and its eigen values are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$. Then, again applying the result (3.1) to the expression on the right hand side of (3.8), we have from (3.7)

$$|X' D_\delta Y Y' D_\delta^{-1} X|^2 \leq |X' D_\delta X| \cdot |X' D^{-1} X| \left(\prod_{i=1}^{k} \lambda_i\right)$$

which proves (3.4).

<u>Proof of (3.5)</u>. Consider the singular value decompositions

$$D^{-1} X X' D = T_1 D_1 T_2' \quad \text{and} \quad D Y Y' D^{-1} = T_3 D_2 T_4'$$

where T_1, T_2, T_3 and T_4 are orthogonal matrices and D_1 and D_2 are diagonal matrices such that $D_1 = \mathrm{Diag}(\beta_1, \ldots, \beta_k, 0, \ldots)$ and $D_2 = \mathrm{Diag}(\gamma_1, \ldots, \gamma_s, 0, \ldots)$, where $\beta_1 \geq \ldots \geq \beta_k > 0$ and $\gamma_1 \geq \ldots \geq \gamma_2 > 0$. Then using a theorem of von Neumann (see equation (2.11) in Rao, 1979),

$$\mathrm{tr}(P_X D_\delta P_Y D_\delta^{-1}) = \mathrm{tr}(D_1 T_2' T_3 D_2 T_4' T_1) \leq \mathrm{tr}\, D_1 D_2 \tag{3.9}$$

and the equality holds iff

$$T_2' T_3 = \begin{pmatrix} I_k & 0 \\ 0 & \Delta_1 \end{pmatrix} \quad \text{and} \quad T_4' T_1 = \begin{pmatrix} I_k & 0 \\ 0 & \Delta_2 \end{pmatrix}$$

where Δ_1 and Δ_2 are arbitrary orthogonal matrices.

Now,

$$\mathrm{tr}\, D_1 D_2 = \sum_{i=1}^{k} \gamma_i \beta_i \leq \left(\sum_{i=1}^{k} \gamma^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} \beta_i^2 \right)^{\frac{1}{2}},$$

$$\sum_{i=1}^{k} \beta_i^2 = \mathrm{tr}(P_X D_\delta P_X D_\delta^{-1}) \tag{3.10}$$

and $\lambda_i = \gamma_i^2$, $i = 1, \ldots, s$ are the nonzero eigen values of $P_Y D_\delta P_Y D_\delta^{-1}$ Hence using (3.10) in (3.9) we get

$$\mathrm{tr}(P_X D_\delta P_Y D_\delta^{-1}) \le (\mathrm{tr}\, P_X D_\delta P_X D_\delta^{-1})^{\frac{1}{2}} (\sum_{i=1}^{k} \lambda_i)^{\frac{1}{2}}$$

which proves (3.5). Thus Lemma 2 is established.

__Lemma 3__. Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_s > 0$ be the eigen values of $Y' D_\delta Y Y' D_\delta^{-1} Y$ where Y is an $n \times s$ matrix such that $Y'Y = I_s$ and diag $D_\delta = (\delta_1, \ldots, \delta_n)$ with all δ_i positive, and $\omega_i = (\delta_i + \delta_{n-i+1})^2 / 4\delta_i \delta_{n-i+1}$ for $i = 1, \ldots, k\,(\le s)$. Then

$$\prod_{i=1}^{k} \lambda_i \le \prod_{i=1}^{k} \omega_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{k} \omega_i. \tag{3.11}$$

__Proof.__ Let

$$\phi(B) = |B'Y' D_\delta Y B| \,/\, |B'(Y' D_\delta^{-1} Y)^{-1} B|$$

$$\Psi(B) = \mathrm{tr}[B'Y' D_\delta Y B\, (B'(Y'D_\delta^{-1} Y)^{-1} B)^{-1}] \tag{3.12}$$

where B is an $s \times k$ matrix with $\rho(B) = k \le s$. Then applying (3.1)

$$\sup_B \phi(B) = \prod_{i=1}^{k} \lambda_i \quad \text{and} \quad \sup_B \Psi(B) \le \sum_{i=1}^{k} \lambda_i. \tag{3.13}$$

Next we observe that

$$B'(Y'D_\delta^{-1}Y)^{-1}B - B'B(B'Y'D_\delta^{-1}YB)^{-1}B'B \tag{3.14}$$

is non-negative definite. (see example 33 on p. 77 of Rao, 1973). Then, substituting the second expression in (3.14) for the first in (3.12), we get

$$\phi(B) \leq |(YB)'D_\delta(YB)| \ |(YB)'D_\delta^{-1}(YB)| / |(YB)'(YB)|^2$$

$$\Psi(B) \leq tr\{(YB)'D_\delta(YB)|(YB)'YB|^{-1}(YB)'D_\delta^{-1}(YB)|(YB)'(YB)|^{-1}. \quad (3.15)$$

Now, writing $YB = L$ which is of rank k and applying the inequalities (1.2) and (1.3) to the right hand sides of (3.15) we have

$$\phi(B) \leq \prod_{i=1}^{k} \omega_i \text{ and } \Psi(B) \leq \sum_{i=1}^{k} \omega_i$$

which in conjunction with (3.13) proves (3.11), since ω_i are independent of B.

Combining the results (3.4) and (3.11) we get

$$|X'D_\delta YY'D_\delta^{-1}X| \leq \left(\prod_{i=1}^{k} \omega_i\right)^{\frac{1}{2}} |X'D_\delta XX'D_\delta^{-1}X|^{\frac{1}{2}}$$

$$tr(X'D_\delta YY'D_\delta^{-1}X) \leq \left(\sum_{i=1}^{k} \omega_i\right)^{\frac{1}{2}} (trX'D_\delta XX'D_\delta^{-1}X)^{\frac{1}{2}}. \quad (3.16)$$

A further application of (1.2) and (1.3) to the right hand side expressions of (3.16) proves Theorem 1.

Proof of Theorem 2. Consider the singular value decomposition, $B = \Delta_1 D_\delta \Delta_2'$ where $\Delta_i' \Delta_i = I_t$ for $i = 1,2$ and $D_\delta = \text{Diag}(\delta_1, \ldots, \delta_t)$ with all δ_i positive. Then $C = B^+ = \Delta_2 D_\delta^{-1} \Delta_1'$. Let $X_o = \Delta_1' X$ and $Y_o = \Delta_2' Y$ so that $X'X - X_o' X_o$ and $Y'Y - Y_o' Y_o$ are n.n.d. matrices and so also

$$P_{Y_o} - Y_o(Y'Y)^{-1}Y_o' \text{ and } P_{X_o} - X_o(X'X)X_o'.$$

Then it is easily seen that

$$\frac{|X' B P_Y C X|^2}{|X'X|^2} \leq \frac{|X_o' \, D_\delta \, P_{Y_o} \, D_\delta^{-1} X_o|^2}{|X_o' \, X_o|^2} \tag{3.17}$$

and

$$tr(P_X B P_Y C) \leq tr(P_{X_o} \, D_\delta \, P_{Y_o} \, D_\delta^{-1}). \tag{3.18}$$

Now using Theorem 1 on the right hand side expressions of (3.17) and (3.18), we get the results (2.7) and (2.8) of Theorem 2.

<u>Proof of Theorem 3</u>. Let V and W be of ranks n_1 and m_1 respectively and write $V = V_1 \, V_1'$ and $W = W_1 \, W_1'$ where V_1 and W_1 are $n \times n_1$ and $m \times m_1$ matrices of ranks n_1 and m_1 respectively. Let

$$B_1 = (V_1' V_1)^{-1} V_1' B W_1 (W_1' W_1)^{-1} \text{ and } C_1 = (W_1' W_1)^{-1} W_1' C V_1 (V_1' V_1)^{-1}.$$

Then, under the given conditions, it is easy to verify that $C_1 = B_1^+$ and taking $X_o = V_1' X$ and $Y_o = W_1' Y$,

$$X' B Y (Y' W Y)^{-1} Y' C X = X_o' B_1 Y_o (Y_o' Y_o)^{-1} Y_o' C_1 X_o' \tag{3.19}$$

and $X' V X = X_o' \, X_o$. Now applying Theorem 2 to the right-hand side expression of (3.19), we get the results of Theorem 3.

REFERENCES

[1] Bloomfield, P. and Watson, G. S. (1975). The inefficiency of least squares. Biometrika **62** 121-128.

[2] Eaton, M. L. (1976). A maximization problem and its applications to canonical correlation. J. Multivar. Anal., **6** 422-425.

[3] Greub, W. and Rheinboldt, W. (1959). On a generalization of an inequality of L. V. Kantorovich. Proc. Amer. Math. Soc., **10** 407-415.

[4] Kantorovich, L. V. (1948). Functional analysis and applied mathematics. Uspehi Mat. Nauk., **3** 89-135; (also translated from the Russian by C. D. Benster, Nat. Bur. Standards Report No. 1509, March 7, 1952).

[5] Khatri, C. G. (1978). Some optimization problems with applications to canonical correlations and sphericity tests. J. Multivar. Anal., **8** 453-467. Corrections. (1980). J. Multivar. Anal., **10**.

[6] Khatri, C. G. and Rao, C. R. (1980). Some extensions of Kantorovich inequality and statistical applications. To appear in J. Multivar. Anal., **10**.

[7] Knott, M. (1975). On the minimum efficiency of least squares. Biometrika **62** 129-132.

[8] Rao, C. R. (1973). Linear Statistical Inference and its Applications. 2nd edition Wiley, New York.

[9] Rao, C. R. (1979). Separation theorems for singular values of matrices and their applications in multivariate analysis. J. Multivar. Anal., **9** 362-377.

[10] Strang, W. G. (1960). On the Kantorovich inequality. Proc. Amer. Math. Soc., **11** 468.